FOURIER TRANSFORM OF LIPSCHITZ FUNCTIONS ON RIEMANNIAN SYMMETRIC SPACES OF RANK ONE

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1 Introduction and statement of main results

Among Riemannian manifolds the symmetric spaces of rank 1 form an important class. On these spaces we can study many problems of geometry, theory of functions and mathematical physics (see [H1], [H2], [H3]). Examples of symmetric spaces of rank 1 are n-dimensional sphere S^n and n-dimensional Lobachevsky space (hyperbolic space) H^n. On symmetric spaces there are analogs of the Fourier series (for compact spaces) and of the Fourier transform (for noncompact spaces), and many problems of classical harmonic analysis have a natural analogue for symmetric spaces. In what follows we will consider only symmetric spaces of noncompact type. Our main result is an analogue of one classical result of E. Titchmarsh connected with the Fourier transform of L^2–functions satisfying certain Lipschitz conditions.

Let $f(x) \in L^2(\mathbb{R})$, $\|\cdot\|_{L^2(\mathbb{R})}$ be the norm on $L^2(\mathbb{R})$, $\alpha \in (0,1)$.

Definition 1. *A function $f(x)$ belongs to the Lipschitz class $Lip(\alpha, 2)$ if*

$$\|f(x+t) - f(x)\|_{L^2(\mathbb{R})} = O(t^\alpha)$$

as $t \to 0$.

Theorem 1 ([T, Theorem 85]). *Let $f(x) \in L^2(\mathbb{R})$ and $\widehat{f}(\lambda)$, $\lambda \in \mathbb{R}$, be the Fourier transform of f. Then the conditions*

$$f \in Lip(\alpha, 2), \quad 0 < \alpha < 1,$$

and

$$\int\limits_{|\lambda| \geqslant r} |\widehat{f}(\lambda)|^2 \, d\lambda = O(r^{-2\alpha})$$

as $r \to \infty$, are equivalent.

Recall some standard definitions connected with symmetric spaces ([H1]). Any Riemannian symmetric space X can be realized as the quotient space G/K, where G is a semisimple connected Lie group with finite center and K is a maximal compact subgroup of G. The group G acts transitively on $X = G/K$ by left translations, and K coincides with the stabilizer of the point $o = eK$ (e is the unity of G). Let $G = NAK$ be the Iwasawa decomposition of G, and let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of the groups G, K, A, N, respectively. We denote by M be the centralizer of the subgroup A in K and put $B = K/M$. Let dx be a G-invariant measure on X; the symbols db and dk will denote the normalized K-invariant measures on B and K, respectively.

We denote by $\mathfrak{a}^*$ the real dual space to $\mathfrak{a}$, and by W the finite Weyl group acting on $\mathfrak{a}^*$. Let Σ be the set of restricted roots ($\Sigma \subset \mathfrak{a}^*$), Σ^+ be the set of restricted positive roots, and

$$\mathfrak{a}^+ = \{h \in \mathfrak{a} : \gamma(h) > 0, \quad \gamma \in \Sigma^+\}$$

be the positive Weyl chamber. Let ρ denote the half-sum of the positive roots (with multiplicity), then $\rho \in \mathfrak{a}^*$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$. This form is positive definite on $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let H_λ denote a vector in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$. The correspondence $\lambda \mapsto H_\lambda$ enables us to identify $\mathfrak{a}^*$ and $\mathfrak{a}$. Via this identification, the action of the Weyl group W can be transferred to $\mathfrak{a}$. Let

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* : H_\lambda \in \mathfrak{a}^+\}.$$

If X is a symmetric space of rank 1, then $\dim \mathfrak{a}^* = 1$, and the set Σ^+ consists of the roots γ and 2γ with some multiplicities m_γ and $m_{2\gamma}$ depending on X (see [H2]). In this case we identify the set $\mathfrak{a}^*$ with $\mathbb{R}$ via the correspondence $\lambda \mapsto \lambda\gamma$, $\lambda \in \mathbb{R}$, and the positive numbers will correspond to the set $\mathfrak{a}^*_+$. The numbers m_γ and $m_{2\gamma}$ often arise in various formulas related to symmetric spaces of rank 1. For instance, the area of the sphere of radius t in X is equal to

$$S(t) = c(\sinh t)^{m_\gamma} (\sinh 2t)^{m_{2\gamma}}, \tag{1.1}$$

where c is a constant, and for the dimension of X we have

$$\dim X = m_\gamma + m_{2\gamma} + 1. \tag{1.2}$$

We return to an arbitrary symmetric space X.

For $g \in G$, let $A(g) \in \mathfrak{a}$ be a unique element for which

$$g = n \cdot \exp A(g) \cdot u,$$

where $u \in K$, $n \in N$. For $x = gK \in X = G/K$ and $b = kM \in B = K/M$ we put

$$A(x, b) := A(k^{-1}g).$$

Let $\mathcal{D}(X)$ and $\mathcal{D}(G)$ denote the sets of infinitely differentiable complex-valued functions with compact support in X and in G, respectively. We note that the functions on $X = G/K$ can be identified in a natural way with the functions $f(g)$ on G satisfying

$$f(gu) = f(g), \quad u \in K.$$

Let dg be the Haar measure on G. We assume that dg is normalized in such a way that

$$\int_X f(x)\, dx = \int_G f(go)\, dg, \quad \forall f \in \mathcal{D}(X), \tag{1.3}$$

where $o = eK \in X = G/K$.

For any functions $f(x) \in \mathcal{D}(X)$, its *Fourier transform*, introduced by S. Helgason [H4], is defined by the formula

$$\widehat{f}(\lambda, b) := \int_X f(x)\, e^{(-i\lambda+\rho)(A(x,b))}\, dx, \quad \lambda \in \mathfrak{a}^*, \quad b \in B = K/M. \tag{1.4}$$

The measure dx on X can be normalized in such a way that the inversion formula for the above Fourier transformation on X will look like this:

$$f(x) = \frac{1}{|W|} \int\limits_{\mathfrak{a}^* \times B} \widehat{f}(\lambda, b)\, e^{(i\lambda+\rho)(A(x,b))} |c(\lambda)|^{-2}\, d\lambda\, db, \tag{1.5}$$

where $|W|$ is the order of the Weyl group, $d\lambda$ is the element of the Euclidean measure on $\mathfrak{a}^*$, and $c(\lambda)$ is the Harish–Chandra function. In what follows, for brevity we put

$$d\mu(\lambda) := |c(\lambda)|^{-2}\, d\lambda.$$

We have the following *Plancherel formula*:

$$\int\limits_X |f(x)|^2\, dx = \frac{1}{|W|} \int\limits_{\mathfrak{a}^* \times B} |\widehat{f}(\lambda, b)|^2\, d\mu(\lambda)\, db = \int\limits_{\mathfrak{a}^*_+ \times B} |\widehat{f}(\lambda, b)|^2\, d\mu(\lambda)\, db. \tag{1.6}$$

By continuity the map $f(x) \mapsto \widehat{f}(\lambda, b)$ extends from $\mathcal{D}(X)$ to an isomorphism of the Hilbert space $L^2(X) = L^2(X, dx)$ onto the Hilbert space $L^2(\mathfrak{a}^*_+ \times B, d\mu(\lambda)\, db)$. This extended map, also denoted $f(x) \mapsto \widehat{f}(\lambda, b)$, keeps the name of the *Fourier transformation*, and the relations (1.5) and (1.6) remain valid.

In what follows, X is a Riemannian symmetric space of noncompact type of rank 1, $n = \dim X$. By $d(x, y)$ we denote distance from x to y, where $x, y \in X$. Let

$$\sigma(x; t) = \{y \in X : d(x, y) = t\}$$

be the sphere in X centered at x of radius $t > 0$. We denote by $d\sigma_x(y)$ the $(n-1)$-dimensional element of area and by $|\sigma(t)|$ the area of entire sphere $\sigma(x; t)$ (the latter quantity is independent of x).

Let $C_c(X)$ be the set of all continuous complex-valued functions on X with compact support. For $f \in C_c(X)$ we define a function $S^t f$ by the formula

$$(S^t f)(x) = \frac{1}{|\sigma(t)|} \int\limits_{\sigma(x;t)} f(y)\, d\sigma_x(y), \quad t > 0, \tag{1.7}$$

The operator S^t is called the shift operator or the spherical averaging operator. It can be proved (see the next section) that the operator S^t extends by continuity from $C_c(X)$ to $L^2(X)$.

Definition 2. *A function $f(x)$ belongs to the Lipschitz class $Lip_X(\alpha, 2)$, $0 < \alpha < 1$, if $f \in L^2(X)$ and*

$$\|S^t f - f\|_{L^2(X)} = O(t^\alpha)$$

as $t \to 0$.

The next theorem is an analogue of Theorem 1 for symmetric spaces.

Theorem 2. *Let X be a Riemannian symmetric space of noncompact type of rank 1, $n = \dim X$. For any function $f(x) \in L^2(X)$ the conditions*

$$f \in Lip_X(\alpha, 2), \quad 0 < \alpha < 1, \tag{1.8}$$

and

$$\int\limits_{|\lambda|\geqslant r}\int\limits_{B}|\widehat{f}(\lambda,b)|^2\,d\lambda\,db=O(r^{-2\alpha-n+1})\tag{1.9}$$

as $r\to+\infty$, *are equivalent.*

The proof of this theorem is the main purpose of the paper. The other analogue of Theorem 1 on the Lobachevsky plane H^2 was considered by Younis [Yo]. This author considered the other shift operator which depends on model of the Lobachevsky plane, whilst the shift S^t has geometrical origin.

2 Auxiliary propositions

Here we collect the auxiliary results of the Harmonic analysis on symmetric spaces.

For any $h\in G$ and $f(x)\in C_c(X)$ we put

$$(T^hf)(x):=\int\limits_{K}f(gkho)\,dk,\tag{2.1}$$

where $x=go$, $g\in G$. If x has another representation $x=g_1o$, $g_1\in G$, then $g_1=g\delta$ for some $\delta\in K$. Using the K-invariance of the Haar measure on K, we obtain

$$\int\limits_{K}f(g_1kho)\,dk=\int\limits_{K}f(g\delta kho)\,dk=\int\limits_{K}f(gkho)\,dk.$$

Consequently, the formula (2.1) is well-defined.

The operator T^h is the other form of the shift operator S^t. It follows from the next lemma.

Lemma1. *If* $h\in G$ *and* $d(ho,o)=t$, *then*

$$(T^hf)(x)=(S^tf)(x),\quad x\in X.\tag{2.2}$$

Proof. For any function $f(x)$ on X and $u\in G$ we put

$$(L_uf)(x):=f(ux).$$

The operators L_u and T^h are commute, that is

$$L_u(T^hf)=T^h(L_uf),\quad f\in C_c(X),\quad u,h\in G.\tag{2.3}$$

Since any element in G is an isometry of X, we have

$$L_u(S^tf)=S^t(L_uf),\quad t>0,\quad u\in G.\tag{2.4}$$

Let $x=go$, $g\in G$. It follows from (2.3) and (2.4) that

$$(T^hf)(x)=(T^hf)(go)=(L_g(T^hf))(o)=(T^h(L_gf))(o),$$
$$(S^tf)(x)=(S^tf)(go)=(L_g(S^tf))(o)=(S^t(L_gf))(o).$$

Therefore, the identity

$$(T^h f)(x) = (S^t f)(x) \quad \forall f \in C_c(X), \quad \forall x \in X,$$

will be proved if we show that

$$(T^h f)(o) = (S^t f)(o) \quad \forall f \in C_c(X). \tag{2.5}$$

Let $C(\sigma(o;t))$ be the set of all continuous functions on the sphere $\sigma(o;t)$. For $\varphi(x) \in C(\sigma(o;t))$ we put

$$I_1(\varphi) := (T^h\varphi)(o), \quad I_2(\varphi) := (S^t\varphi)(o),$$

assuming that $\varphi(x)$ is extended in some way from $\sigma(o;t)$ to a function of class $C_c(X)$; the values of $I_1(\varphi)$ and $I_2(\varphi)$ do not depend on specific way of extension. Obviously, I_1 and I_2 are positive linear functionals invariant with respect to the action of K, that is,

$$I_1(L_k\varphi) = I_1(\varphi), \quad I_2(L_k\varphi) = I_2(\varphi) \quad \forall k \in K.$$

The functionals I_1 and I_2 give rise to K-invariant measures on the sphere $\sigma(o;t)$. Since any symmetric space of rank 1 is a two-point homogeneous manifold (see [W]), it follows that K acts transitively on $\sigma(o;t)$. Hence, a K-invariant measure on $\sigma(o;t)$ is uniquely determined up to a coefficient. If we take $\varphi_0(x) \equiv 1$, then

$$I_1(\varphi_0) = I_2(\varphi_0) = 1;$$

therefore, I_1 and I_2 coincide. This implies (2.5) and (2.2).

Let $L^p(X) = L^p(X, dx)$, $1 \leqslant p < \infty$, and $\|\cdot\|_p$ be the norm in the Banach space $L^p(X)$.

Lemma 2. *For any function $f \in C_c(X)$ and any $g \in G$ we have*

$$\|T^h f\|_p \leqslant \|f\|_p. \tag{2.6}$$

Proof. Let $p > 1$. If $\varphi(k)$ is a continuous function on the group K then using $\int_K dk = 1$ and the Hölder inequality, we obtain

$$\left|\int_K \varphi(k)\, dk\right|^p \leqslant \int_K |\varphi(k)|^p\, dk. \tag{2.7}$$

The inequality (2.7) is obvious for the case $p = 1$.

Let $f(x) \in C_c(X)$. Using (1.3) and (2.7) we obtain

$$\begin{aligned}\|T^h f\|_p^p &= \int_X |T^h f(x)|^p\, dx = \int_G |T^h f(go)|^p\, dg = \\ &= \int_G \left| \int_K f(gkho)\, dk\right|^p dg \leqslant \int_G \int_K |f(gkho)|^p\, dk\, dg = \\ &= \int_K \left(\int_G |f(gkho)|^p\, dg\right) dk = \int_K \left(\int_G |f(go)|^p\, dg\right) dk = \|f\|_p^p\,.\end{aligned}$$

Here we have used the invariance of the Haar measure dg on G with respect to the right shifts.

Since $C_c(X)$ is dense in $L^p(X)$ for $1 \leqslant p < \infty$, Lemma 2 shows that the operator T^h (and the operator S^t) can be extended to a continuous operator on the entire space $L^p(X)$, and (2.3) remains true for any $f \in L^p(X)$. In particular

$$\|S^t f\|_2 \leqslant \|f\|_2, \quad f \in L^2(X). \tag{2.8}$$

In harmonic analysis on symmetric spaces the central role plays spherical functions (see [H2], [GW], [ST]). For $\lambda \in \mathfrak{a}^*$, let $\varphi_\lambda(g)$ denote the spherical function on G defined by the Harish–Chandra formula

$$\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))}\, dk, \quad g \in G, \tag{2.9}$$

where all notations is defined in §1. We list some properties of the spherical functions to be used later on:

$$\varphi_\lambda(u_1 g u_2) = \varphi_\lambda(g), \quad u_1, u_2 \in K; \tag{2.10}$$

$$\varphi_\lambda(g^{-1}) = \varphi_\lambda(g), \quad g \in G; \tag{2.11}$$

$$\varphi_\lambda(e) = 1. \tag{2.12}$$

Lemma 3. *Let $\Phi(f)(\lambda, b) := \widehat{f}(\lambda, b)$ be the Fourier transform of $f(x) \in L^2(X)$. Then*

$$\Phi(T^h f)(\lambda, b) = \varphi_\lambda(h) \cdot \widehat{f}(\lambda, b), \quad h \in G. \tag{2.13}$$

Proof. Since $\mathcal{D}(X)$ is dense in $L^2(X)$, it is sufficient to prove (2.13) for $f \in \mathcal{D}(X)$. We recall that the element $A(g) \in \mathfrak{a}$ is defined from the Iwasawa decomposition

$$g = n \cdot \exp A(g) \cdot u, \tag{2.14}$$

where $u \in K$, $n \in N$. Let $g, h \in G$, $k \in K$. Since

$$kgh = n \cdot \exp(A(kg)) \cdot u(kg) \cdot h$$

for $n \in N$, $u(kg) \in K$, and since the subgroup A involved in an Iwasawa decomposition normalizes the subgroup N, we obtain

$$A(kgh) = A(kg) + A(u(kg)h). \tag{2.15}$$

For brevity we put

$$e_\lambda(g) := e^{(-i\lambda+\rho)(A(g))}, \quad g \in G.$$

From definition of the Fourier transformation we deduce that

$$\widehat{T^h f}(\lambda, b) = \int\limits_G \left(\int\limits_K f(gvho)\, dv \right) e_\lambda(k^{-1}g)\, dg, \tag{2.16}$$

where $b = kM$, $k \in K$. Since $uo = o$ and $A(gu) = A(g)$ for $u \in K$, the right-hand side in (2.16) can be reshaped as follows:

$$\widehat{T^h f}(\lambda, b) = \int_K \int_G f(gvhv^{-1}o)\, e_\lambda(k^{-1}g)\, dg\, dv =$$

$$= \int_K \int_G f(go)\, e_\lambda(k^{-1}gvh^{-1}v^{-1})\, dg\, dv =$$

$$= \int_G f(go) \Big(\int_K e_\lambda(k^{-1}gvh^{-1})\, dv \Big)\, dg. \tag{2.17}$$

Then, by (2.15) we have

$$e_\lambda(k^{-1}gvh^{-1}) = e_\lambda(k^{-1}g) \cdot e_\lambda(uvh^{-1}),$$

where $u = u(k^{-1}g)$. Substituting this in (2.17), we obtain

$$\widehat{T^h f}(\lambda, b) = \int_G f(go)\, e_\lambda(k^{-1}g)\, \big(\int_K e_\lambda(uvh^{-1})\, dv \big)\, dg =$$

$$= \varphi_\lambda(h^{-1})\, \widehat{f}(\lambda, b) = \varphi_\lambda(h) \cdot \widehat{f}(\lambda, b).$$

This completes the proof.

Let h_1 and h_2 be elements of the group G such that $d(h_1 o, o) = d(h_2 o, o) = t$. Since X is a two-point homogeneous space (see [W]), it follows that K acts transitively on $\sigma(o; t)$. Hence, $h_2 o = uh_1 o$ for some $u \in K$. Since K is the stabilizer of the point o, it follows from $h_2 o = uh_1 o$ that $h_2 = uh_1 v$ for some $v \in K$. By (2.10) we have $\varphi_\lambda(h_1) = \varphi_\lambda(h_2)$, consequently, the function $\varphi_\lambda(h)$ depends only on the distance $t = d(ho, o)$, and we will write often $\varphi_\lambda(t)$ instead of $\varphi_\lambda(h)$ for $t = d(ho, o)$.

Using Lemma 1 we can rewrite (2.13) in the form

$$\Phi(S^t f)(\lambda, b) = \varphi_\lambda(t) \cdot \widehat{f}(\lambda, b), \quad t \in \mathbb{R}_+ = [0; +\infty). \tag{2.18}$$

We recall that the set $\mathfrak{a}^*$ is identified with $\mathbb{R}$ (see §1). Since $\varphi_\lambda(t) = \varphi_{-\lambda}(t)$, we will propose $\lambda \in \mathbb{R}_+$.

Lemma 4 (Estimates for spherical functions) *For any spherical function $\varphi_\lambda(t)$, $\lambda \geqslant 0$, $t \geqslant 0$, we have:*
1) $|\varphi_\lambda(t)| \leqslant 1$;
2) $1 - \varphi_\lambda(t) \leqslant t^2(\lambda^2 + \rho^2)$;
3) *there is a constant $c > 0$ depending only on X such that if $\lambda t \geqslant 1$, then*

$$1 - \varphi_\lambda(t) \geqslant c. \tag{2.19}$$

Proof. See [Pl, Lemmas 3.1, 3.2 and 3.3].

3 Proof of Theorem 2

Proof of implication (1.8) $\Rightarrow$ (1.9)

Let $f(x) \in L^2(X)$ and the condition (1.8) is true, that is

$$\|S^t f - f\|_2 = O(t^\alpha) \tag{3.1}$$

as $t \to 0$. From the Plancherel formula and (2.18) it follows that

$$\|S^t f - f\|_2^2 = \int_X |S^t f(x) - f(x)|^2 \, dx =$$

$$= \int_0^\infty \int_B |1 - \varphi_\lambda(t)|^2 \, |\widehat{f}(\lambda, b)|^2 \, d\mu(\lambda) \, db. \tag{3.2}$$

For brevity we denote

$$F(\lambda) := \int_B |\widehat{f}(\lambda, b)|^2 \, db. \tag{3.3}$$

It follows from (3.2) and (3.3) that we can rewrite (3.1) in the form

$$\int_0^\infty |1 - \varphi_\lambda(t)|^2 \, F(\lambda) \, d\mu(\lambda) = O(t^{2\alpha}). \tag{3.4}$$

We recall that

$$d\mu(\lambda) = |c(\lambda)|^{-2} \, d\lambda,$$

where $c(\lambda)$ is the Harish–Chandra function of the symmetric space X. For any functions $A(\lambda) > 0$ and $B(\lambda) > 0$ we say that

$$A(\lambda) \asymp B(\lambda)$$

as $\lambda \to \infty$ if

$$c_1 \, B(\lambda) \leqslant A(\lambda) \leqslant c_2 \, B(\lambda)$$

for some constants $c_1 > 0$ and $c_2 > 0$. Below, $c, c_1, c_2, \ldots$ are positive constants that may depend on X and α and independent on f and λ.

The Harish–Chandra function $c(\lambda)$ can be expressed in terms of the Γ-function of Euler (see [H2, Chapter 4, §6]). When X is a symmetric space of rank 1, we have

$$(c(\lambda))^{-1} = c_0 \frac{\Gamma(\frac{1}{4}m_\gamma + \frac{1}{2} + \frac{1}{2}\lambda) \, \Gamma(\frac{1}{4}m_\gamma + \frac{1}{2}m_{2\gamma} + \frac{1}{2}\lambda)}{\Gamma(\frac{1}{2}\lambda + \frac{1}{2}) \, \Gamma(\frac{1}{2}\lambda)}, \tag{3.5}$$

where $c_0 > 0$ is a constant, m_γ and $m_{2\gamma}$ are the multiplicities of the roots γ and 2γ in Σ^+ (see §1).

Using well known limiting relation

$$\lim_{\lambda \to +\infty} \frac{\Gamma(\lambda + a)}{\Gamma(\lambda) \, \lambda^a} = 1$$

(see [BE]) we get from (3.5) that

$$|c(\lambda)|^{-2} \asymp \lambda^{m_\gamma + m_{2\gamma}}.$$

Since $m_\gamma + m_{2\gamma} + 1 = n = \dim X$, it follows that

$$|c(\lambda)|^{-2} \asymp \lambda^{n-1}, \quad n = \dim X. \tag{3.6}$$

If $\lambda \in \left[\frac{1}{t}, \frac{2}{t}\right]$ then $\lambda t \geqslant 1$, and we get from (2.19) that

$$1 \leqslant \frac{1}{c^2} |1 - \varphi_\lambda(t)|^2. \tag{3.7}$$

Then

$$\int_{1/t}^{2/t} F(\lambda)\, d\mu(\lambda) \leqslant \frac{1}{c^2} \int_{1/t}^{2/t} |1 - \varphi_\lambda(t)|^2 F(\lambda)\, d\mu(\lambda) \leqslant$$
$$\leqslant \frac{1}{c^2} \int_0^\infty |1 - \varphi_\lambda(t)|^2 F(\lambda)\, d\mu(\lambda) = O(t^{2\alpha}). \tag{3.8}$$

It follows from (3.8) and (3.6) that

$$\int_{1/t}^{2/t} F(\lambda)\, \lambda^{n-1}\, d\lambda = O(t^{2\alpha}) \tag{3.9}$$

as $t \to 0$ or, equivalently,

$$\int_r^{2r} F(\lambda)\, \lambda^{n-1}\, d\lambda = O(r^{-2\alpha}) \tag{3.10}$$

as $r \to +\infty$. We can rewrite (3.10) in the equivalent form

$$\int_r^{2r} F(\lambda)\, d\lambda = O(r^{-2\alpha - n + 1}) \tag{3.11}$$

as $r \to +\infty$.

It follows from (3.11) that

$$\int_r^{2r} F(\lambda)\, d\lambda \leqslant c_1\, r^{-2\alpha - n + 1}, \tag{3.12}$$

where $c_1 > 0$ is a constant. Using this inequality, we get

$$\int_r^\infty F(\lambda)\, d\lambda = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1} r} F(\lambda)\, d\lambda \leqslant \sum_{k=0}^\infty c_1\, (2^k r)^{-2\alpha - n + 1} \leqslant c_2\, r^{-2\alpha - n + 1},$$

and the condition (1.9) is true. We obtain that (1.8) implies (1.9).

Proof of implication (1.9) $\Rightarrow$ (1.8)

Let $f(x) \in L^2(X)$ and the condition (1.9) is true, that is

$$\int_r^\infty F(\lambda)\,d\lambda = O(r^{-2\alpha-n+1}) \tag{3.13}$$

as $r \to \infty$. It follows from (3.13) that

$$\int_r^{2r} F(\lambda)\,d\lambda = O(r^{-2\alpha-n+1}),$$

hence,

$$\int_r^{2r} F(\lambda)\,\lambda^{n-1}\,d\lambda \leqslant 2^{n-1} r^{n-1} \int_r^{2r} F(\lambda)\,d\lambda \leqslant c_3\, r^{-2\alpha}.$$

Now

$$\int_r^\infty F(\lambda)\,\lambda^{n-1}\,d\lambda = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} F(\lambda)\,\lambda^{n-1}\,d\lambda \leqslant c_3 \sum_{k=0}^\infty 2^{-2\alpha k} r^{-2\alpha} \leqslant c_4 r^{-2\alpha}.$$

Hence

$$\int_r^\infty F(\lambda)\,\lambda^{n-1}\,d\lambda = O(r^{-2\alpha})$$

and, by (3.6),

$$\int_r^\infty F(\lambda)\,d\mu(\lambda) = O(r^{-2\alpha}). \tag{3.14}$$

We can rewrite (3.2) in the form

$$\|S^t f - f\|_2^2 = I_1 + I_2, \tag{3.15}$$

where

$$I_1 = \int_0^{1/t} |1 - \varphi_\lambda(t)|^2 F(\lambda)\,d\mu(\lambda), \tag{3.16}$$

$$I_2 = \int_{1/t}^\infty |1 - \varphi_\lambda(t)|^2 F(\lambda)\,d\mu(\lambda). \tag{3.17}$$

Let us obtain the upper bounds of I_1 and I_2. Using (3.14) and $|\varphi_\lambda(t)| \leqslant 1$, we get

$$I_2 = \int_{1/t}^\infty |1 - \varphi_\lambda(t)|^2 F(\lambda)\,d\mu(\lambda) \leqslant 4 \int_{1/t}^\infty F(\lambda)\,d\mu(\lambda) = O(t^{2\alpha}). \tag{3.18}$$

Using the inequalities 1) and 2) of Lemma 4, we obtain

$$I_1 = \int_0^{1/t} |1-\varphi_\lambda(t)|^2 F(\lambda)\, d\mu(\lambda) \leqslant 2\int_0^{1/t} |1-\varphi_\lambda(t)| F(\lambda)\, d\mu(\lambda) \leqslant$$

$$\leqslant 2t^2 \int_0^{1/t} (\lambda^2+\rho^2)\, F(\lambda)\, d\mu(\lambda) = I_3 + I_4, \tag{3.19}$$

where

$$I_3 = 2\rho^2 t^2 \int_0^{1/t} F(\lambda)\, d\mu(\lambda),$$

$$I_4 = 2t^2 \int_0^{1/t} \lambda^2\, F(\lambda)\, d\mu(\lambda).$$

Using the Plancherel formula, we get

$$I_3 \leqslant 2\rho^2 t^2 \int_0^{\infty} F(\lambda)\, d\mu(\lambda) = 2\rho^2 t^2 \|f\|_2^2 = O(t^{2\alpha}), \tag{3.20}$$

since $2\alpha < 2$.

Temporarily we denote

$$\psi(r) := \int_r^{\infty} F(\lambda)\, d\mu(\lambda).$$

Using integration by parts, we get

$$I_4 = 2t^2 \int_0^{1/t} (-r^2\psi'(r))\, dr = 2t^2\left(-\frac{1}{t^2}\psi\left(\frac{1}{t}\right) + 2\int_0^{1/t} r\,\psi(r)\, dr\right) =$$

$$= -2\psi\left(\frac{1}{t}\right) + 4t^2 \int_0^{1/t} r\,\psi(r)\, dr.$$

Since $\psi(r) = O(r^{-2\alpha})$ (see (3.14)), we have $r\,\psi(r) = O(r^{1-2\alpha})$ and

$$\int_0^{1/t} r\,\psi(r)\, dr = O\left(\int_0^{1/t} r^{1-2\alpha}\, dr\right) = O(t^{2\alpha-2}).$$

Hence,

$$I_4 = O(t^{2\alpha}). \tag{3.21}$$

Finally, by (3.15), (3.18), (3.19), (3.20) and (3.21), we obtain

$$\|S^t f - f\|_2^2 = O(t^{2\alpha})$$

as $t \to 0$, i.e. (1.9) implies (1.8). This completes the proof of Theorem 2.

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